Proof of Pythagorean and law of cosines

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The proof of the Pythagorean theorem is given based on basic geometric arguments. The proof is extended to the more general law of cosines.

Proof of Pythagorean theorem

Based in geometric considerations, we are going to give proof for one of the most widely known identities in mathematics –the Pythagorean theorem. It states that the sides of a right triangle (that is, a triangle for which one of the inner angles measures 90° or $\pi/2$ radians) of lengths a, b and c satisfy the following identity:

$$c^2 = a^2 + b^2. \tag{1}$$

Consider the triangle of sides a, b and c marked in bold of Fig. 1. If we construct the system of triangles shown in the figure, then it is clear that the area of the bigger square is $(a+b)^2$. The area of all the elements contained within must of course equal $(a+b)^2$. There are four triangles of area $\frac{1}{2}ab$ and one square of area c^2. Hence, it must be

$$(a+b)^2 = 4\frac{1}{2}ab + c^2. \tag{2}$$

Expanding the parenthesis in Eq. (2) we have

$$a^2 + b^2 + \cancel{2ab} = \cancel{2ab} + c^2, \tag{3}$$

or

$$a^2 + b^2 = c^2, \tag{4}$$

as we wanted to prove.

Law of cosines

The Pythagorean theorem is only valid for right triangles. There is a more general theorem, the *law of cosines*,

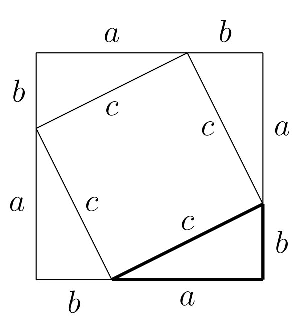

FIG. 1. Geometrical construction made to prove the Pythagorean theorem.

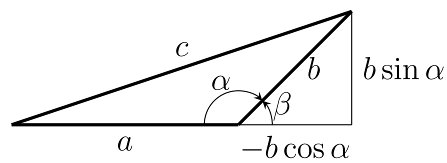

FIG. 2. Non right triangle to which the law of cosines applies.

that relates the lengths of the sides of any given triangle as

$$c^2 = a^2 + b^2 - 2ab\cos\alpha, \tag{5}$$

where α is the angle formed by the sides of lengths a and b, and opposite c, as shown in Fig. 2. In that figure, the side of length a is extended to form a right triangle that will help prove the identity given by Eq. (5).

It is clear from Fig. 2 that $\beta = \pi - \alpha$, in radians, or $\beta = 180° - \alpha$ in degrees. Here we prefer to work in radians, and so will use the former expression. Although usually it would suffice to say that

$$\begin{aligned} \cos\beta &= \cos(\pi - \alpha) = -\cos\alpha \qquad \text{and} \\ \sin\beta &= \sin(\pi - \alpha) = \sin\alpha, \end{aligned} \tag{6}$$

we are trying to make as few assumptions as possible. Therefore the geometric proof to Eq. (6) is given in Fig. 3. Having constructed a right triangle in Fig. 2 we can now

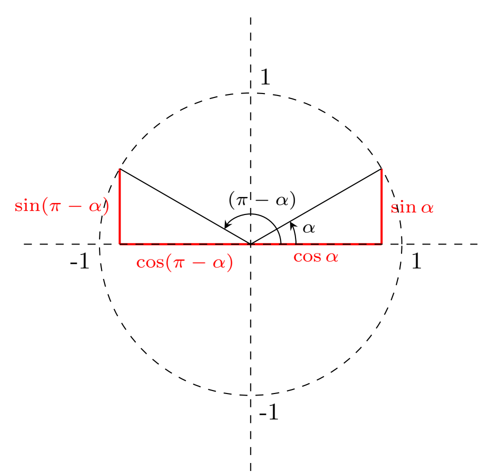

FIG. 3. Geometric proof that $\sin(\pi-\alpha) = \sin\alpha$ and $\cos(\pi-\alpha) = -\cos\alpha$.

relate the lenth of c to the other sides of the bigger triangle through the Pythagorean theorem:

$$c^2 = (a - b\cos\alpha)^2 + (b\sin\alpha)^2 . \tag{7}$$

Expanding the powers in parenthesis, one easily gets to

$$c^2 = a^2 + b^2\cos^2\alpha - 2ab\cos\alpha + b^2\sin^2\alpha. \tag{8}$$

Given the well known trigonometric relation $\cos^2\alpha + \sin^2\alpha = 1$,[1] Eq. (8) reduces to

$$c^2 = a^2 + b^2 - 2ab\cos\alpha, \tag{9}$$

which is the relation we wanted to prove.

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[1] The relation $\cos^2\alpha + \sin^2\alpha = 1$ can easily be proven using the Pythagorean theorem for a triangle for which $c = 1$ and α is the angle between a and c. It follows immediately that $a = \cos\alpha$ and $b = \sin\alpha$.